

SEMISTABILITY VS. NEFNESS FOR (HIGGS) VECTOR BUNDLES

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ABSTRACT

Generalizing a result of Miyaoka, we prove that the semistability of a vector bundle E on a smooth projective curve over a field of characteristic zero is equivalent to the nefness of any of certain divisorial classes θ_s , λ_s in the Grassmannians $\text{Gr}_s(E)$ of locally-free quotients of E and in the projective bundles $\mathbb{P}Q_s$, respectively (here $0 < s < \text{rk } E$ and Q_s is the universal quotient bundle on $\text{Gr}_s(E)$). The result is extended to Higgs bundles. In that case a necessary and sufficient condition for semistability is that all classes λ_s are nef. We also extend this result to higher-dimensional complex projective varieties by showing that the nefness of the classes λ_s is equivalent to the semistability of the bundle E together with the vanishing of the characteristic class $\Delta(E) = c_2(E) - \frac{r-1}{2r}c_1(E)^2$.

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1. INTRODUCTION

Let E be a vector bundle on a smooth projective curve C over a field of characteristic 0. According to Miyaoka [7], the semistability of E is equivalent to the fact that a suitable divisor λ in the projective bundle $\mathbb{P}E$ is numerically effective (nef) [7]. The numerical class λ is defined as

$$\lambda = [c_1(\mathcal{O}_{\mathbb{P}E}(1))] - \frac{1}{r}(\deg E) F$$

where $r = \operatorname{rk} E$ and F is class of the fibre of the projection $\mathbb{P}E \rightarrow C$.

A mild generalization of Miyaoka's criterion is the following. We consider the Grassmannians $\operatorname{Gr}_s(E)$ of rank s locally-free quotients of E and the projectivized universal quotient bundles $\mathbb{P}Q_s$ on them. Moreover we introduce the classes

$$\theta_s = c_1(Q_s) - \frac{s}{r} \deg(E) F_s$$

in $\operatorname{Gr}_s(E)$ and

$$\lambda_s = c_1(\mathcal{O}_{\mathbb{P}Q_s}(1)) - \mu(E) F_s$$

in $\mathbb{P}Q_s$ (in both cases F_s denotes the class of the fibre of the projection onto C). We prove:

Theorem 1.1. *If E is semistable, all classes θ_s and λ_s are nef. Conversely, if one of these classes is nef (for just one value of s), E is semistable.*

The interest in this result is that it suggests a generalization of Miyaoka's criterion to the case of Higgs bundles (E, ϕ) on a smooth projective curve C . (Throughout this paper we shall use the notation \mathfrak{E} for a Higgs bundle (E, ϕ) .) We introduce schemes $\mathfrak{Gr}_s(\mathfrak{E})$ parametrizing locally-free rank s Higgs quotients of \mathfrak{E} (i.e., locally-free quotients of E whose kernels are ϕ -invariant), and the universal quotient Higgs bundles $\mathfrak{Q}_s = (Q_s, \Phi_s)$ on them. We define classes $\theta_{s, \mathfrak{E}}$ in $\mathfrak{Gr}_s(\mathfrak{E})$ and $\lambda_{s, \mathfrak{E}}$ in $\mathbb{P}Q_s$ as in the previous case.

Theorem 1.2. *If \mathfrak{E} is semistable, all classes $\theta_{s, \mathfrak{E}}$ and $\lambda_{s, \mathfrak{E}}$ are nef. Conversely, if all classes $\lambda_{s, \mathfrak{E}}$ are nef, then \mathfrak{E} is semistable.*

It is indeed not enough that one of the classes $\lambda_{s, \mathfrak{E}}$ is nef to ensure the semistability of \mathfrak{E} , as we show in Section 3.

In Example 3.9 we apply this criterion to the Higgs bundle whose semistability implies the Miyaoka inequality for the Chern classes of a projective surface with ample canonical bundle.

The last generalization we give is about Higgs bundles on a complex projective manifold X of any dimension. (We need to restrict to the complex case as we use transcendental techniques.) In this case we define the classes

$$\lambda_{s,\mathfrak{E}} = c_1(\mathcal{O}_{\mathbb{P}Q_s}(1)) - \frac{1}{r}\pi_s^*c_1(E)$$

where $\pi_s: \mathfrak{Gr}_s(\mathfrak{E}) \rightarrow X$ is the projection. Let $\Delta(E)$ be the characteristic class

$$\Delta(E) = c_2(E) - \frac{r-1}{2r}c_1(E)^2 = \frac{1}{2r}c_2(E \otimes E^*).$$

We prove the following result.

Theorem 1.3. *Let \mathfrak{E} be a Higgs bundle on a complex projective manifold. The following conditions are equivalent.*

- i) *All classes $\lambda_{s,\mathfrak{E}}$ are nef, for $0 < s < r$.*
- ii) *\mathfrak{E} is semistable and $\Delta(E) = 0$.*

Note that the classes $\lambda_{s,\mathfrak{E}}$ do not depend on the choice of a polarization in X , so when they are all nef the Higgs bundle \mathfrak{E} is semistable with respect to all polarizations. Conversely, if \mathfrak{E} is semistable with respect to a given polarization, and $\Delta(E) = 0$, then \mathfrak{E} is semistable with respect to *all* polarizations. Our results also imply that if all classes $\lambda_{s,\mathfrak{E}}$ are nef, then \mathfrak{E} is semistable after restriction to *any* smooth projective curve in X .

Theorem 1.3 applies also to ordinary bundles in the following form:

Theorem 1.4. *Let E be a vector bundle on a complex projective manifold. The following conditions are equivalent.*

- i) *The class λ_1 is nef.*
- ii) *E is semistable and $\Delta(E) = 0$.*

This result was already contained in [1] as a special case and is proved by repeating verbatim the proof of Theorem 1.3 (see Section 4).

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2. SEMISTABILITY VS. NEFNESS FOR VECTOR BUNDLES

All varieties we shall consider will be over an algebraically closed field of characteristic 0. Let X be a smooth projective variety, E a holomorphic vector bundle on it, and denote by $\mathbb{P}E$ the projectivization of E , defined as

$$\mathbb{P}E = \mathbf{Proj}(\mathcal{S}(E)),$$

where $\mathcal{S}(E)$ is the symmetric algebra of the sheaf of sections of E . We recall that a bundle E is said to be *ample* if the hyperplane line bundle $\mathcal{O}_{\mathbb{P}E}(1)$ on $\mathbb{P}E$ is ample [4, 5]. A weaker notion is that of *numerical effectiveness*: a bundle E is said to be numerically effective (nef) if the class $c_1(\mathcal{O}_{\mathbb{P}E}(1))$ is numerically effective. If both E and E^* are numerically effective, then E is said to be *numerically flat*. The following result has been proved in [2].

Proposition 2.1. *All Chern classes of a numerically flat bundle vanish.* \square

We recall that given a smooth projective variety X with a choice of a polarization H , a torsion-free coherent sheaf \mathcal{E} on X is said to be *semistable* (in Mumford-Takemoto's sense) if for every proper coherent subsheaf \mathcal{F} of \mathcal{E} one has

$$\mu(\mathcal{F}) \leq \mu(\mathcal{E}),$$

where the slope $\mu(\mathcal{E})$ of a torsion-free coherent sheaf is defined as

$$\mu(\mathcal{E}) = \frac{\deg \mathcal{E}}{\mathrm{rk} \mathcal{E}}, \quad \deg \mathcal{E} = c_1(\mathcal{E}) \cdot H^{n-1}$$

if $n = \dim X$. If the inequality always holds strictly, the bundle E is said to be *stable*.

Following Miyaoka [7], we introduce some notation and state the theorem relating the semistability of a vector bundle to the nefness of a suitable divisorial class.

Definition 2.2. *For every smooth projective variety X , one denotes:*

- (i) $N^1(X) = \frac{\mathrm{Pic}(X)}{\text{num. eq.}} \otimes \mathbb{R}$;
- (ii) $NA(X) \subset N^1(X)$ the ample cone of X (the cone generated by the classes of ample divisors), and $\overline{NA}(X)$ its closure (the set of classes of nef divisors of X);
- (iii) $N_1(X) = \frac{A_1(X)}{\text{num. eq.}} \otimes \mathbb{R}$;
- (iv) $NE(X) \subset N_1(X)$ the real cone generated by the effective 1-cycles.

Theorem 2.3. ([7], Theorem 3.1.) *Let $\pi: E \rightarrow C$ be a rank r vector bundle on a smooth projective curve C , and define the class in $N^1(\mathbb{P}E)$*

$$\lambda = [c_1(\mathcal{O}_{\mathbb{P}E}(1))] - \mu(E) F$$

where F is the class in $N^1(\mathbb{P}E)$ of the fibre of the projection $\mathbb{P}E \rightarrow C$. Then the following conditions are equivalent:

- (i) E is semistable;
- (ii) λ is nef;
- (iii) $\overline{NA}(\mathbb{P}E) = \mathbb{R}_+ \lambda + \mathbb{R}_+ F$;
- (iv) $\overline{NE}(\mathbb{P}E) = \mathbb{R}_+ \lambda^{r-1} + \mathbb{R}_+ \lambda^{r-2} \cdot F$;
- (v) every effective divisor in $\mathbb{P}E$ is nef. □

(Here \mathbb{R}_+ is the set of nonnegative real numbers.) The class $r\lambda$ is the relative anti-canonical class of $\mathbb{P}E \rightarrow C$, and one has $\lambda^r = 0$.

Proof. For the readers' convenience, and following [7], we include a sketch of a proof of this theorem.

(i) \Rightarrow (ii). If λ is not nef there is an irreducible curve $C' \subset \mathbb{P}E$ such that $[C'] \cdot \lambda < 0$. After a suitable base change $f: C'' \rightarrow C$ we may assume that C' is a union of sections C_j of the bundle $\mathbb{P}(f^*E)$, and $[C_j] \cdot \lambda' < 0$ for all j , where λ' is the class λ for the bundle $\pi'': \mathbb{P}(f^*E) \rightarrow C''$. There are surjections $f^*E \rightarrow \pi''_* \mathcal{O}_{C_j}(1)$, and $\deg(\mathcal{O}_{C_j}(1)) = [C_j] \cdot \lambda' + \mu(f^*E) < \mu(f^*E)$. But this contradicts the semistability of E (note that since the morphism f is separable and finite, the bundle E is semistable if and only if f^*E is, cf. [7, Prop. 3.2] and our Lemma 3.3).

(ii) \Rightarrow (iv). If Γ is a class in $\overline{NE}(\mathbb{P}E)$, one has $\Gamma = a\lambda^{r-1} + b\lambda^{r-2} \cdot F$ with $a \geq 0$. Since λ is nef, one has $b = \Gamma \cdot \lambda \geq 0$.

(iii) and (iv) are easily shown to be equivalent.

(iii) and (iv) \Rightarrow (v). Let $D = a\lambda + bF$ be an effective divisor class. All 1-cycles $D \cdot (\lambda + \varepsilon F)^{r-2}$ lie in $\overline{NE}(\mathbb{P}E)$ for every positive real number ε , and so do their limits $D \cdot \lambda^{r-2}$. Then $a, b \geq 0$ by (iv) and D is nef by (iii).

(v) \Rightarrow (i). Let F be a destabilizing subbundle of E and let $\alpha \in \mathbb{Q}$ be such that $\mu(F) > \alpha > \mu(E)$. For N big enough the space

$$\begin{aligned} H^0(\mathrm{Sym}^N F(-N\alpha p)) &\subset H^0(C, \mathrm{Sym}^N E(-N\alpha p)) \\ &\simeq H^0(\mathbb{P}E, \mathcal{O}_{\mathbb{P}E}(N) \otimes \pi^* \mathcal{O}_C(-N\alpha p)) \end{aligned}$$

(where p is a point in C) is nonempty; therefore, the class $N(\lambda + (\mu(E) - \alpha)F)$ is effective but not nef. \square

We describe now the first generalization of this result. Given a vector bundle E on an algebraic variety X , we shall denote by $\text{Gr}_s(E)$ the Grassmann variety of rank s locally-free quotients of E , with $0 < s < r = \text{rk } E$. We have a morphism $p_s: \text{Gr}_s(E) \rightarrow X$ that makes $\text{Gr}_s(E)$ a bundle of Grassmannians. On every variety $\text{Gr}_s(E)$ a universal quotient bundle Q_s is defined, in a such a way that for any morphism $f: Y \rightarrow X$ and any rank s locally-free quotient F of f^*E , there is a morphism $\psi_F: Y \rightarrow \text{Gr}_s(E)$ over X (that is, $f = p_s \circ \psi_F$) such that $F = \psi_F^* Q_s$.

Let θ_s be the class in $N^1(\text{Gr}_s(E))$

$$\theta_s = [c_1(Q_s)] - \frac{s}{r} \deg(E) F_s,$$

where F_s is the class of the fibre of the projection $\pi_s: \text{Gr}_s(E) \rightarrow X$.

Theorem 2.4. *If E is a semistable vector bundle on a smooth projective curve C , the class θ_s is nef for every s , $0 < s < r = \text{rk } E$.*

Proof. This result can be proved according to the lines of the proof of the implication (i) \Rightarrow (ii) in Theorem 2.3. Alternatively, one can use the Plücker embedding $\varpi: \text{Gr}_s(E) \rightarrow \mathbb{P}(\Lambda^s E)$ to reduce to the case of a projective bundle, as we now show. The morphism ϖ embeds into a commutative diagram

$$\begin{array}{ccc} \mathbb{P}Q_s & \xrightarrow{\quad} & \mathbb{P}(\Lambda^s E) \\ \downarrow & \nearrow \varpi & \downarrow \pi_\Lambda \\ \text{Gr}_s(E) & \xrightarrow{p_s} & C \end{array}$$

and one has an isomorphism

$$(1) \quad \varpi^* \mathcal{O}_{\mathbb{P}(\Lambda^s E)}(1) \simeq \det Q_s.$$

The induced morphisms

$$\varpi^*: N^1(\mathbb{P}(\Lambda^s E)) \rightarrow N^1(\text{Gr}_s(E)), \quad \varpi_*: N_1(\text{Gr}_s(E)) \rightarrow N_1(\mathbb{P}(\Lambda^s E))$$

are isomorphisms, as one easily shows by using some Schubert calculus.

If E is semistable, so is $\Lambda^s E$ for every s , $0 < s < r$ [6], so that the class

$$\lambda_{\Lambda^s E} = [c_1(\mathcal{O}_{\mathbb{P}(\Lambda^s E)}(1))] - \mu(\Lambda^s E) [\pi_\Lambda^*(x)]$$

is nef. By restricting to the image of the Plücker embedding one obtains that θ_s is nef. \square

The converse to Theorem 2.4 is as follows.

Theorem 2.5. *If for some s (with $0 < s < r = \text{rk } E$) the class θ_s is nef, then E is semistable.*

Proof. If $\theta_s \in N^1(\text{Gr}_s(E))$ is nef, the class $\lambda_{\Lambda^s E} = (\varpi^*)^{-1}(\theta_s)$ is nef as well, since for any curve $\Gamma \subset \mathbb{P}(\Lambda^s E)$ one has $\lambda_{\Lambda^s E} \cdot [\Gamma] = \theta_s \cdot [\Gamma']$ with $\varpi(\Gamma') = \Gamma \cap \varpi(\text{Gr}_s(E))$. By Miyaoka's result the bundle $\Lambda^s E$ is semistable. It is then an easy task to prove that E is semistable as well. \square

Remark 2.6. These constructions provide an alternative algebraic proof of the fact that, given a semistable bundle E on a smooth projective variety X , its exterior powers $\Lambda^s E$ are semistable as well. By the Metha-Ramanathan theorem (cf. e.g. [7]), it is enough to consider the case when X is a curve. Then by the proof of Theorem 2.4 which follows [7] we know that the classes θ_s are nef, so that the classes $\lambda_{\Lambda^s E}$ are nef, whence $\Lambda^s E$ is semistable. \square

We consider now another construction. Again, E is a rank r vector bundle on a smooth projective curve C , $\text{Gr}_s(E)$ is the Grassmannian bundle of its rank s quotients, and Q_s the universal quotient bundle on $\text{Gr}_s(E)$. We define the class in $N^1(\mathbb{P}Q_s)$

$$\lambda_s = [c_1(\mathcal{O}_{\mathbb{P}Q_s}(1))] - \mu(E) F_s$$

where F_s is the class of the fibre of the composition $\mathbb{P}Q_s \rightarrow \text{Gr}_s(E) \rightarrow C$.

Theorem 2.7. *If E is semistable, the class λ_s is nef for every s , $0 < s < r = \text{rk } E$.* \square

To prove this result we need a Lemma.

Lemma 2.8. *Let G be a semistable vector bundle on a smooth projective curve C , and let C' be an irreducible curve in $\mathbb{P}G$. Denote by ξ the class of $\mathcal{O}_{\mathbb{P}G}(1)$ in $N^1(\mathbb{P}G)$. Then,*

$$[C'] \cdot \xi \geq \mu(G) p_*[C']$$

where $p: \mathbb{P}G \rightarrow C$ is the projection.

Proof. By Theorem 2.3 we have

$$[C'] = a\lambda^{r-1} + b\lambda^{r-2} \cdot F$$

(here $\lambda = [c_1(\mathcal{O}_{\mathbb{P}G}(1)) - \mu(G)F]$ and $r = \text{rk } G$) with $a, b \geq 0$, so that

$$[C'] \cdot \xi = (a\lambda^{r-1} + b\lambda^{r-2}F)(\lambda + \mu(G)F) = a\mu(G) + b \geq a\mu(G).$$

Moreover, one has $a = p_*[C']$. □

Proof of Theorem 2.7. If for some s the class λ_s is not nef there is an irreducible curve $C' \subset \mathbb{P}Q_s$ which surjects onto C and is such that $C' \cdot \lambda_s < 0$. Let $h: C'' \rightarrow C$ be a finite morphism and consider the commutative diagram whose squares are cartesian

$$\begin{array}{ccc} \mathbb{P}Q'_s & \xrightarrow{h''} & \mathbb{P}Q_s \\ \pi'_s \downarrow & & \downarrow \pi_s \\ \text{Gr}_s(h^*E) & \xrightarrow{h'} & \text{Gr}_s(E) \\ p'_s \downarrow & & \downarrow p_s \\ C''' & \xrightarrow{h} & C \end{array}$$

We may choose the pair (C'', h) in such a way that the fibre product $\tilde{C} = C'' \times_C C'$ (a curve in $\mathbb{P}Q'_s$) is a union of curves C_j which project onto C''' with degree one (and meet the fibre F'_s at just one point). One has $[C_j] \cdot \lambda'_s < 0$. Let Γ_j be the projection of C_j onto $\text{Gr}_s(h^*E)$, denote by Q_j the restriction of Q'_s to it, and let $E_j = (h'^* \circ p_s^* E)|_{\Gamma_j}$. We have an epimorphism $E_j \rightarrow Q_j \rightarrow 0$. The composition $h \circ p'_s$ restricted to Γ_j (call it h_j) is a finite morphism (actually, an isomorphism), so that E_j is semistable. Now with the help of Lemma 2.8 we have:

$$\begin{aligned} \mu(Q_j) &\leq \frac{[C_j] \cdot \xi_j}{\pi'_{s*}[C_j]} = [C_j] \cdot \xi'_s \\ &= [C_j] \cdot (\lambda'_s + \mu(h^*E) F'_s) < \mu(E_j) \end{aligned}$$

but this contradicts the semistability of E_j . □

Corollary 2.9. *If E is a semistable bundle of positive (resp. nonnegative) degree on a smooth projective curve C , then all universal quotient bundles Q_s are ample (resp. numerically effective).*

Proof. This result may be proved by mimicking Gieseker's proof for $s = 1$, cf. [3]. □

Theorem 2.10. *If for some s (with $0 < s < r = \text{rk } E$) the class λ_s is nef, then E is semistable.*

Proof. By direct computation one sees that $\pi_{s*}(\lambda_s)^s = \theta_s$. So, if Γ is a curve in $\text{Gr}_s(E)$, one has $[\Gamma] \cdot \theta_s = (\lambda_s)^s \cdot \pi_s^*[\Gamma] \geq 0$ (since λ_s is nef) so that θ_s is nef, whence E is semistable. \square

3. SEMISTABILITY VS. NEFNESS FOR HIGGS BUNDLES

We want to investigate if the semistability of a Higgs bundle can be encoded in the nefness of some suitable classes. In particular, we prove Theorem 1.2.

3.1. Grassmannians of Higgs quotients. We recall the basic definitions about Higgs bundles (cf. [9], [10]).

Definition 3.1. *Let X be a projective variety. A Higgs sheaf \mathfrak{E} on X is a coherent sheaf E on X endowed with a morphism $\phi: E \rightarrow E \otimes \Omega_X$ of \mathcal{O}_X -modules such that $\phi \wedge \phi = 0$, where Ω_X is the cotangent sheaf to X . A Higgs subsheaf F of a Higgs sheaf $\mathfrak{E} = (E, \phi)$ is a subsheaf of E such that $\phi(F) \subset F \otimes \Omega_X$. A Higgs bundle is a Higgs sheaf \mathfrak{E} such that E is a locally-free \mathcal{O}_X -module.*

Definition 3.2. *Let X be a smooth projective variety equipped with a polarization. A Higgs sheaf $\mathfrak{E} = (E, \phi)$ is semistable (resp. stable) if it is torsion-free, and $\mu(F) \leq \mu(E)$ (resp. $\mu(F) < \mu(E)$) for every proper nontrivial Higgs subsheaf F of \mathfrak{E} .*

In the sequel we shall need the following Lemma. It generalizes a well-known fact about semistable vector bundles that we have already used in this paper [3, 7].

Lemma 3.3. *Let $f: Y \rightarrow X$ be a finite separable morphism of smooth projective curves, \mathfrak{E} a Higgs bundle on X and $f^*\mathfrak{E}$ the pullback Higgs bundle on Y . Then \mathfrak{E} is semistable if and only if $f^*\mathfrak{E}$ is semistable.*

Proof. The “if” part is straightforward. To prove that $f^*\mathfrak{E}$ is semistable on Y when \mathfrak{E} is semistable on X we can assume that f is a Galois covering. Let $0 \rightarrow F \rightarrow f^*E$ be a maximal destabilizing Higgs subbundle. For every element σ in the Galois group of f , σ^*F is also a maximal destabilizing Higgs subbundle of $\sigma^*E = E$ so that $\sigma^*F = F$ by the uniqueness of F . It follows that $F = f^*E'$ for a certain subbundle $0 \rightarrow E' \rightarrow E$. Since E' destabilizes E , we have only to prove that it is a Higgs subbundle of \mathfrak{E} , i.e., that the composed morphism $E' \xrightarrow{\phi|_{E'}} E \otimes \Omega_X \rightarrow (E/E') \otimes \Omega_X$ vanishes. Since f is faithfully flat,

we may as well prove that the induced morphism $F \xrightarrow{f^*\phi} f^*E \otimes f^*\Omega_X \rightarrow (f^*E/F) \otimes f^*\Omega_X$ vanishes. But this follows from the diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 F & \xrightarrow{f^*\phi} & f^*E \otimes f^*\Omega_X & \longrightarrow & (f^*E/F) \otimes f^*\Omega_X & & \\
 & \searrow \phi_Y & \downarrow & & \downarrow & & \\
 0 \longrightarrow & F \otimes \Omega_Y & \longrightarrow & f^*E \otimes \Omega_Y & \longrightarrow & (f^*E/F) \otimes \Omega_Y & \longrightarrow 0
 \end{array}$$

since $\phi_Y: F \rightarrow f^*E \otimes \Omega_Y$ takes values in $F \otimes \Omega_Y$. \square

Given a Higgs sheaf \mathfrak{E} , we may construct the closed subschemes $\mathfrak{Gr}_s(\mathfrak{E}) \subset \text{Gr}_s(E)$ parametrizing the rank s locally-free Higgs quotients, i.e. locally-free quotients of E such that the corresponding kernels are ϕ -invariant. This can be done as follows: let us consider the universal exact sequence

$$(2) \quad 0 \rightarrow S_{r-s} \xrightarrow{\psi} p_s^*E \xrightarrow{\eta} Q_s \rightarrow 0$$

of sheaves on the Grassmannian $\text{Gr}_s(E)$ that defines the universal quotient bundle Q_s . Then $\mathfrak{Gr}_s(\mathfrak{E})$ is the closed subvariety of $\text{Gr}_s(E)$ where the composed morphism

$$(\eta \otimes 1) \circ p_s^*(\phi) \circ \psi: S_{r-s} \rightarrow Q_s \otimes p_s^*\Omega_X$$

vanishes (the equations for $\mathfrak{Gr}_s(\mathfrak{E})$ inside $\text{Gr}_s(E)$ are written in subsection 3.2 for the special case $s = 1$). We denote by $\pi_{s,\mathfrak{E}}$ the projections $\mathfrak{Gr}_s(\mathfrak{E}) \rightarrow X$. The restriction of (2) to the scheme $\mathfrak{Gr}_s(\mathfrak{E})$ gives a new universal exact sequence

$$0 \rightarrow S_{r-s} \xrightarrow{\psi} (\pi_{s,\mathfrak{E}})^*E \xrightarrow{\eta} Q_{s,\mathfrak{E}} \rightarrow 0$$

and $Q_{s,\mathfrak{E}}$ is a rank s universal Higgs quotient vector bundle. This means that for every morphism $f: Y \rightarrow X$ and every rank s Higgs quotient F of f^*E there is a morphism $\psi_F: Y \rightarrow \mathfrak{Gr}_s(\mathfrak{E})$ such that $f = \pi_{s,\mathfrak{E}} \circ \psi_F$ and $F = \psi_F^*Q_{s,\mathfrak{E}}$. Note that the kernel S_{r-s} of the morphism $(\pi_{s,\mathfrak{E}})^*E \rightarrow Q_{s,\mathfrak{E}}$ is ϕ -invariant.

For every s we define the classes

$$\theta_{s,\mathfrak{E}} \in N^1(\mathfrak{Gr}_s(\mathfrak{E})), \quad \lambda_{s,\mathfrak{E}} \in N^1(\mathbb{P}Q_{s,\mathfrak{E}})$$

as in the previous Section. We have:

Theorem 3.4. *If \mathfrak{E} is a semistable Higgs bundle on a smooth projective curve C , all classes $\theta_{s,\mathfrak{E}}$ and $\lambda_{s,\mathfrak{E}}$ are nef.*

Proof. A possible proof of the nefness of $\theta_{s,\mathfrak{E}}$ runs as in the proof of Theorem 2.4 which follows [7]. Analogously, the proof of the nefness of $\lambda_{s,\mathfrak{E}}$ runs as in the proof of Theorem 2.7. \square

Remark 3.5. The proof of Theorem 2.7 adapted to the case of Higgs bundles shows that if $\lambda_{s,\mathfrak{E}}$ is not nef, after a base change \mathfrak{E} is destabilized by a rank s locally free quotient.

Corollary 3.6. *If \mathfrak{E} is a semistable Higgs bundle on a smooth projective curve C of positive (resp. nonnegative) degree, then for all s the universal quotient bundle Q_s is ample (resp. numerically effective).*

Proof. One again adapts the proof by Gieseker in [3], this time using Lemma 3.3. \square

3.2. Equations of the scheme of rank-one Higgs quotients. For some time we concentrate on the scheme $\mathfrak{Gr}_1(\mathfrak{E})$ of rank one Higgs quotients of a rank r Higgs vector bundle \mathfrak{E} on a n -dimensional smooth variety X . We have a closed immersion $j: \mathfrak{Gr}_1(\mathfrak{E}) \hookrightarrow \mathbb{P}E$, and the universal Higgs quotient is $Q_1 = \mathcal{O}_{\mathfrak{Gr}_1(\mathfrak{E})}(1) = j^*\mathcal{O}_{\mathbb{P}E}(1)$. We denote by $\pi_1: \mathfrak{Gr}_1(\mathfrak{E}) \rightarrow X$ the projection.

We denote by $\xi_E = c_1(\mathcal{O}_{\mathbb{P}E}(1))$ the hyperplane class in $\mathbb{P}E$ and write $\xi_{\mathfrak{E}} = c_1(\mathcal{O}_{\mathfrak{Gr}_1(\mathfrak{E})}(1)) = j^*(\xi_E)$. We also define

$$\lambda_{\mathfrak{E}} = \left[\xi_{\mathfrak{E}} - \frac{1}{r} \pi_1^* c_1 E \right] \in N^1(\mathfrak{Gr}_1(\mathfrak{E})).$$

One can write local equations for $\mathfrak{Gr}_1(\mathfrak{E})$ by using the Euler sequence

$$0 \rightarrow \Omega_{\mathbb{P}(E)/X} \otimes \mathcal{O}_{\mathbb{P}E}(1) \xrightarrow{\psi} \pi^* E \xrightarrow{\eta} \mathcal{O}_{\mathbb{P}E}(1) \rightarrow 0,$$

because this is the form that (2) takes in this case. We then know that $\mathfrak{Gr}_1(\mathfrak{E})$ is the closed set where the composition of morphisms

$$(\eta \otimes 1) \circ \pi^* \phi \circ \psi: \Omega_{\mathbb{P}(E)/X} \otimes \mathcal{O}_{\mathbb{P}E}(1) \rightarrow \mathcal{O}_{\mathbb{P}E}(1) \otimes \pi^* \Omega_X^1$$

vanishes. Given a local basis of sections (e_1, \dots, e_r) of E , which can be taken as local vertical homogeneous coordinates for $\mathbb{P}E$, the Higgs field is represented by a matrix $(\phi_{\alpha\beta})$ of 1-forms by letting $\phi(e_\beta) = \sum_{\alpha} \phi_{\alpha\beta} e_\alpha$. The homogeneous equations for $\mathfrak{Gr}_1(\mathfrak{E})$ are

$$(3) \quad \sum_{\gamma} e_{\gamma} (\phi_{\gamma\beta} e_{\alpha} - \phi_{\gamma\alpha} e_{\beta}) = 0, \text{ for every } 1 \leq \alpha < \beta \leq r.$$

So $\mathfrak{Gr}_1(\mathfrak{E})$ is locally the intersection of $n \binom{r}{2}$ hyperquadrics in $\mathbb{P}E$. Let us study this locus in the case when the Higgs bundle \mathfrak{E} is *nilpotent*, i.e., there is a decomposition

$$E = E_1 \oplus \cdots \oplus E_m$$

as a direct sum of subbundles, and $\phi(E_i) \subseteq E_{i+1} \otimes \Omega_X$ for $1 \leq i < m$, $\phi(E_m) = 0$.

The induced morphism $\phi \otimes 1: E \otimes \Omega_X^* \rightarrow E$ yields a Higgs quotient sheaf $\mathfrak{Q} = (Q, 0)$ of \mathfrak{E} , where $Q = \text{coker}(\phi \otimes 1)$, and there is also a Higgs quotient bundle $\bar{\mathfrak{E}} = (E/E_m, \bar{\phi})$, where $\bar{\phi}$ is the Higgs field induced by ϕ . There are closed immersions

$$\mathfrak{Gr}_1(\mathfrak{Q}) = \mathbb{P}(Q) \hookrightarrow \mathfrak{Gr}_1(\mathfrak{E}), \quad \mathfrak{Gr}_1(\bar{\mathfrak{E}}) \hookrightarrow \mathfrak{Gr}_1(\mathfrak{E}).$$

The homogeneous equations for $\mathbb{P}(Q)$ are given locally by the images of a basis of $(\phi \otimes 1)(E \otimes \Omega_X^*)$, that is, by

$$(4) \quad \sum_{\beta} \phi_{\alpha\beta} e_{\beta} = 0 \quad \text{for every } e_{\alpha} \in E_i, i < m.$$

Proposition 3.7. *The scheme of rank one Higgs quotients is the closed subscheme of $\mathbb{P}(E)$ given by*

$$\mathfrak{Gr}_1(\mathfrak{E}) = \mathfrak{Gr}_1(\bar{\mathfrak{E}}) \cup \mathfrak{Gr}_1(\mathfrak{Q}) \cup Z.$$

where $Z \subseteq \mathfrak{Gr}_1(\bar{\mathfrak{E}}) \cap \mathfrak{Gr}_1(\mathfrak{Q})$ is a union of embedded components.

Proof. We can proceed locally. Let us write $n_i = \dim(E_1 \oplus \cdots \oplus E_i)$ and take for every subbundle E_i a local basis $\{e_{\gamma}\}$ of sections with $n_{i-1} < \gamma \leq n_i$. If we consider the subset of the equations (3) where $e_{\beta} \in E_m$ and $e_{\alpha} \in E_i$ for $i < m$, we get

$$0 = e_{\beta} \cdot \sum_{n_i < \gamma \leq n_{i+1}} e_{\gamma} \phi_{\gamma\alpha},$$

for every pair (α, β) as above. These equations describe the locus $Y \cup Y'$, where

$$Y' \equiv \{e_{\beta} = 0 \mid \text{for every } e_{\beta} \in E_m\} \equiv \mathbb{P}(E/E_m)$$

$$Y \equiv \left\{ \sum_{n_i < \gamma \leq n_{i+1}} e_{\gamma} \phi_{\gamma\alpha} = 0 \mid \text{for every } e_{\alpha} \in E_i, i < m \right\} \equiv \mathbb{P}(Q),$$

where the last equality is due to (4). The remaining equations are

$$0 = e_{\alpha} \cdot \left(\sum_{n_j < \gamma \leq n_{j+1}} e_{\gamma} \phi_{\gamma\beta} \right) - e_{\beta} \cdot \left(\sum_{n_i < \gamma \leq n_{i+1}} e_{\gamma} \phi_{\gamma\alpha} \right)$$

for $e_{\alpha} \in E_i$, $e_{\beta} \in E_j$, $i \leq j < m$ (and $\alpha < \beta$ if $i = j$) and define hyperquadrics containing Y .

If $m = 2$, the only possibility is $i = j = 1 = m - 1$ so that the e_γ 's in the equations above belong to E_m , thus proving that the corresponding hyperquadric contains also Y' . Moreover, $\bar{\phi} = 0$ in this case, so that $Y' = \mathfrak{Gr}_1(\bar{\mathfrak{E}})$ and we conclude that

$$\mathfrak{Gr}_1(\mathfrak{E}) = \mathfrak{Gr}_1(\bar{\mathfrak{E}}) \cup \mathfrak{Gr}_1(\Omega) = \mathbb{P}(E/E_2) \cup \mathbb{P}(Q)$$

when $m = 2$.

Assume now that $m \geq 3$. Since the points in $Y \equiv \mathbb{P}(Q)$ satisfy the equations (3), we have that

$$\mathfrak{Gr}_1(\mathfrak{E}) = (\mathfrak{Gr}_1(\mathfrak{E}) \cap Y') \cup \mathbb{P}(Q)$$

Moreover, the equations for $\mathfrak{Gr}_1(\mathfrak{E}) \cap Y'$ are the equations (3) for $j \leq m - 1$, which are easily shown to be the equations for $\mathfrak{Gr}_1(\bar{\mathfrak{E}})$. Then $\mathfrak{Gr}_1(\mathfrak{E}) \cap Y' \equiv \mathfrak{Gr}_1(\bar{\mathfrak{E}})$ and $\mathfrak{Gr}_1(\mathfrak{E}) = \mathfrak{Gr}_1(\bar{\mathfrak{E}}) \cup \mathbb{P}(Q)$ up to a union of embedded components. \square

3.3. Equations in the case of curves. Let \mathfrak{E} be a nilpotent Higgs bundle on a smooth projective curve C and denote by $\widetilde{\mathfrak{Gr}}_1(\mathfrak{E})$ the union of all components of $\mathfrak{Gr}_1(\mathfrak{E})$ not contained in a fibre of $\mathbb{P}(E) \rightarrow C$. Similar meaning will have the expressions $\widetilde{\mathfrak{Gr}}_1(\Omega)$ or $\widetilde{\mathfrak{Gr}}_1(\bar{\mathfrak{E}})$. The symbol $\bar{\lambda}_{1,\mathfrak{E}}$ will denote the restriction of $\lambda_{1,\mathfrak{E}}$ to $\widetilde{\mathfrak{Gr}}_1(\mathfrak{E})$.

Proposition 3.8. *The class of $\widetilde{\mathfrak{Gr}}_1(\mathfrak{E})$ in the Chow ring of $\mathbb{P}(E)$ is*

$$\begin{aligned} [\widetilde{\mathfrak{Gr}}_1(\mathfrak{E})] &= [\widetilde{\mathfrak{Gr}}_1(\Omega)] + j_*[\widetilde{\mathfrak{Gr}}_1(\bar{\mathfrak{E}})] \\ &= \xi^{r-r(\phi)} - [\deg(\phi(E)) + r(\phi)(2 - 2g(C)) + \deg(T(Q))]\xi^{r-r(\phi)-1} \cdot F \\ &\quad + j_*[\widetilde{\mathfrak{Gr}}_1(\bar{\mathfrak{E}})] \end{aligned}$$

where $r(\phi) = \text{rk}(\phi(E))$, $j: \mathbb{P}(E/E_m) \hookrightarrow \mathbb{P}(E)$ is the natural immersion and $T(Q)$ is the torsion subsheaf of Q .

Proof. We start by computing the class of $\mathbb{P}(G)$ where

$$0 \rightarrow N \rightarrow E \rightarrow G \rightarrow 0$$

is a quotient rank q bundle. One has

$$[\mathbb{P}(G)] = a\xi^{r-q} + b\xi^{r-q-1} \cdot F$$

where ξ is the relative hyperplane class of $\mathbb{P}(E)$, F is the class of a fibre of π and a, b are integer numbers. Since $1 = \xi_G^{q-1} \cdot F_G = \xi^{q-1} \cdot F \cdot [\mathbb{P}(G)]$ we obtain $a = 1$. Moreover

$\xi^r = \pi^*(c_1(E)) \cdot \xi^{r-1} = r\mu(E)F \cdot \xi^{r-1} = r\mu(E)$ and similarly $\xi_G^q = q\mu(G)$, and we get $b = q\mu(G) - r\mu(E)$, that is

$$(5) \quad \begin{aligned} [\mathbb{P}(G)] &= \xi^{r-q} + (q\mu(G) - r\mu(E))\xi^{r-q-1} \cdot F \\ &= \xi^{r-q} - (\deg(E) - \deg(G))\xi^{r-q-1} \cdot F = \xi^{r-q} - \deg(N)\xi^{r-q-1} \cdot F. \end{aligned}$$

When G has torsion $T(G)$, we actually have $\widetilde{\mathfrak{Gr}}_1(\Omega) = \mathbb{P}(G/T(G))$. Since $G/T(G)$ is a quotient vector bundle, we can compute as above to get

$$(6) \quad [\widetilde{\mathfrak{Gr}}_1(\Omega)] = [\mathbb{P}(G/T(G))] = \xi^{r-q} - (\deg(N) + \deg(T(G)))\xi^{r-q-1} \cdot F.$$

This formula implies our claim. \square

3.4. Unstable Higgs bundles \mathfrak{E} such that $\tilde{\lambda}_{1,\mathfrak{E}}$ is nef. Let \mathfrak{E} be a rank three nilpotent Higgs bundle on a smooth projective curve C , having the form $E = L_1 \oplus L_2 \oplus L_3$ where each L_i is a line bundle and $\phi(L_1) \subseteq L_2 \otimes \Omega_C$, $\phi(L_2) \subseteq L_3 \otimes \Omega_C$, $\phi(L_3) = 0$.

Let us write $\alpha_i = c_1(L_i)$. If we impose that the Higgs subbundles L_3 and $L_2 \oplus L_3$ do not destabilize \mathfrak{E} we obtain the inequalities

$$(7) \quad \alpha_1 + \alpha_2 - 2\alpha_3 \geq 0 \quad \text{and}$$

$$(8) \quad 2\alpha_1 - \alpha_2 - \alpha_3 \geq 0$$

One can prove that these inequalities are actually sufficient for \mathfrak{E} to be semistable.

Now we study when the restrictions of $\tilde{\lambda}_{\mathfrak{E}}$ to the components of $\widetilde{\mathfrak{Gr}}_1(\mathfrak{E})$ are nef. There are two components,

$$\widetilde{\mathfrak{Gr}}_1(\mathfrak{E}) = \widetilde{\mathfrak{Gr}}_1(\bar{\mathfrak{E}}) \cup \widetilde{\mathfrak{Gr}}_1(\Omega)$$

with $\widetilde{\mathfrak{Gr}}_1(\bar{\mathfrak{E}}) \simeq \widetilde{\mathfrak{Gr}}_1(\Omega) \simeq C$. Here $\bar{\mathfrak{E}}$ is the Higgs bundle given by E/L_3 with the induced Higgs morphism. Let us write $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ for the restrictions of $\tilde{\lambda}_{1,\mathfrak{E}}$ to each of the components of $\widetilde{\mathfrak{Gr}}_1(\mathfrak{E})$.

Since $E/L_3 \simeq L_1 \oplus L_2$, by the nilpotent rank two case we have

$$\begin{aligned} [\widetilde{\mathfrak{Gr}}_1(\bar{\mathfrak{E}})] &= 2(\xi - \alpha_2 F) \cdot [\mathbb{P}(L_1 \oplus L_2)] \\ &= 2(\xi - \alpha_2 F)(\xi - \alpha_3 F) \\ &= 2(\xi^2 - (\alpha_2 + \alpha_3)\xi \cdot F), \end{aligned}$$

so that

$$\tilde{\lambda}_1 = 2(\xi^2 - (\alpha_2 + \alpha_3)\xi \cdot F)(\xi - \frac{1}{3}(\alpha_1 + \alpha_2 + \alpha_3)F) = \frac{2}{3}(2\alpha_1 - \alpha_2 - \alpha_3).$$

On the other hand, since

$$Q = L_1 \oplus (L_2/(\phi \otimes 1)(L_1 \otimes \Omega_C^*)) \oplus (L_3/(\phi \otimes 1)(L_2 \otimes \Omega_C^*))$$

by modding the torsion out we obtain $Q/T(Q) = L_1$ and

$$[\widetilde{\mathfrak{Gr}_1(\mathfrak{Q})}] = [\mathbb{P}(L_1)] = \xi^2 - (\alpha_2 + \alpha_3)\xi \cdot F,$$

by Eq (6). Then

$$\begin{aligned} \tilde{\lambda}_2 &= (\xi^2 - (\alpha_2 + \alpha_3)\xi \cdot F)(\xi - \frac{1}{3}(\alpha_1 + \alpha_2 + \alpha_3)F) \\ &= \frac{1}{3}(2\alpha_1 - \alpha_2 - \alpha_3). \end{aligned}$$

So $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ are nef if and only if inequality (8) holds, and if that inequality holds, $\tilde{\lambda}_{1,\mathfrak{E}}$ is nef.

Let C be a smooth projective curve of genus 2, and $K = x + y$ a canonical divisor. Let us consider the line bundles

$$L_1 = \mathcal{O}_C(K + x), \quad L_2 = \mathcal{O}_C(x), \quad L_3 = \mathcal{O}_C(3x)$$

Since $L_2 \otimes L_1^{-1} \otimes \Omega_C = \mathcal{O}_C$, there exists a nonzero morphism $\phi_{21}: L_1 \rightarrow L_2 \otimes \Omega_C$. Moreover, $L_3 \otimes L_2^{-1} \otimes \Omega_C = \mathcal{O}_C(2x + K)$ so that there is also a nonzero morphism $\phi_{32}: L_2 \rightarrow L_3 \otimes \Omega_C$. We can then define a nilpotent Higgs field $\phi: E \rightarrow E \otimes \Omega_C$ on

$$E = L_1 \oplus L_2 \oplus L_3$$

as being equal to ϕ_{21} on L_1 , to ϕ_{32} on L_2 and zero on L_3 . Now,

$$2\alpha_1 - \alpha_2 - \alpha_3 = 2,$$

that is, the inequality (8) is true, so that $\tilde{\lambda}_{1,\mathfrak{E}}$ is nef. The restriction of the class $\lambda_{1,\mathfrak{E}}$ to a component of $\mathfrak{Gr}_1(\mathfrak{E})$ lying in a fibre of $\mathbb{P}E \rightarrow C$ coincides with the restriction of the class $\xi_E = [c_1(\mathcal{O}_{\mathbb{P}E}(1))]$, hence is nef, and the class $\lambda_{1,\mathfrak{E}}$ itself is nef. However,

$$\alpha_1 + \alpha_2 - 2\alpha_3 = -2,$$

so that the inequality (7) does not hold, and \mathfrak{E} is not semistable.

It is interesting to check in this example what happens with the class $\theta_{2,\mathfrak{E}}$ in $\mathfrak{Gr}_2(\mathfrak{E})$. If we again remove the components embedded in fibres of $\mathbb{P}E \rightarrow C$, we obtain a subscheme $\widetilde{\mathfrak{Gr}_2(\mathfrak{E})} \simeq \mathbb{P}L_3 \simeq C$, and the class $\tilde{\theta}_{2,\mathfrak{E}}$ (the restriction of $\theta_{2,\mathfrak{E}}$ in to $\widetilde{\mathfrak{Gr}_2(\mathfrak{E})}$) is

$$\tilde{\theta}_{2,\mathfrak{E}} = \frac{1}{3}(\alpha_1 + \alpha_2 - 2\alpha_3) < 0.$$

Then $\tilde{\theta}_{2,\mathfrak{E}}$ is not nef, so that $\theta_{2,\mathfrak{E}}$ is not nef either, and the class $\lambda_{2,\mathfrak{E}}$ is in turn not nef, for the argument contained in the proof of Theorem 2.10.

3.5. Conclusion of the proof of Theorem 1.2. We need to show that if all classes $\lambda_{s,\mathfrak{E}} \in N^1(\mathbb{P}Q_{s,\mathfrak{E}})$ are nef, then \mathfrak{E} is semistable.

Let $\mathfrak{E}' = (E', \phi)$ be a rank s locally-free Higgs quotient of \mathfrak{E} . Then there is a section $\sigma: C \rightarrow \mathfrak{Gr}_s(\mathfrak{E})$ such that $E' = \sigma^*Q_s$. Consider the curve $C_\sigma = \sigma(C) \subset \mathfrak{Gr}_s(\mathfrak{E})$, the restriction $Q_s^\sigma = Q_{s|C_\sigma}$ and the class $\bar{\lambda}_s = \lambda_{s,\mathfrak{E}|_{\mathbb{P}Q_s^\sigma}}$. Since $\lambda_{s,\mathfrak{E}}$ is nef by hypothesis, the class $\bar{\lambda}_s$ is nef as well. On the other hand, we have

$$\bar{\lambda}_s = \lambda_{\mathbb{P}Q_s^\sigma} + (\mu(E') - \mu(E))F_s^\sigma$$

where F_s^σ is the class of the fibre of the projection $\mathbb{P}Q_s^\sigma \rightarrow C_\sigma$, and

$$\lambda_{\mathbb{P}Q_s^\sigma} = [c_1(\mathcal{O}_{\mathbb{P}Q_s^\sigma}(1))] - \mu(Q_s^\sigma) F_s^\sigma$$

(note that $\mu(Q_s^\sigma) = \mu(E')$). Since $(\lambda_{\mathbb{P}Q_s^\sigma})^s = 0$ and $(\lambda_{\mathbb{P}Q_s^\sigma})^{s-1} \cdot F_s^\sigma = 1$, the condition $(\bar{\lambda}_s)^s \geq 0$ implies $\mu(E') \geq \mu(E)$, so that \mathfrak{E} is semistable.

Example 3.9. Let X be a smooth projective surface over \mathbb{C} with ample canonical class K . As an application of the criterion established in Theorem 1.2 we prove the semistability of the Higgs bundle $F = \Omega_X \oplus \mathcal{O}_X$, with Higgs structure given by the morphism ϕ (cf. [8])

$$\begin{aligned} \Omega_X \oplus \mathcal{O}_X &\rightarrow (\Omega_X \otimes \Omega_X) \oplus \Omega_X \\ (\omega, f) &\mapsto (0, \omega). \end{aligned}$$

The interest of this example is that since (F, ϕ) is semistable it satisfies the Bogomolov inequality (which holds true also for semistable Higgs bundles, cf. [9]), which in this case yields the Miyaoka-Yau inequality $3c_2(X) \geq c_1(X)^2$.

Since K is ample, X admits a Kähler-Einstein metric [11], hence the cotangent bundle Ω_X is semistable with respect to the polarization K . Let C be a curve in the linear system $|mK|$, with m big enough for $\Omega = \Omega_{X|C}$ to be semistable. Let $E = F|_C = \Omega \oplus \mathcal{O}_C$. It is sufficient to prove that the Higgs bundle \mathfrak{E} is semistable.

By analyzing the possible rank-1 locally-free Higgs quotients of E one finds that $\mathfrak{Gr}_1(\mathfrak{E})$ has two components, one isomorphic to C , with $Q_1 \simeq \Omega_C$ and $\lambda_{1,\mathfrak{E}} = c_1(\Omega_C)$, which is nef; the other component is isomorphic to $\mathbb{P}\Omega$ with $Q_1 = \mathcal{O}_{\mathbb{P}\Omega}(1)$ and $\lambda_{1,\mathfrak{E}} = \lambda_{\mathbb{P}\Omega}$, which is nef because Ω is semistable.

For rank-2 locally-free Higgs quotients we find that $\mathfrak{Gr}_2(\mathfrak{E})$ has two components both isomorphic to C . In one case $Q_2 \simeq \Omega$ and $\lambda_{2,\mathfrak{E}} = c_1(\Omega)$, and in the other $Q_2 \simeq \Omega_C \oplus \mathcal{O}_C$ with $\lambda_{2,\mathfrak{E}} = K_C$, which is nef. So \mathfrak{E} is semistable.

4. THE HIGHER-DIMENSIONAL CASE

In this section we prove Theorem 1.3. In extending the semistability criterion to the higher-dimensional case we shall use transcendental techniques. So we assume that X is a projective n -dimensional smooth variety over \mathbb{C} with a choice of a polarization H . Let $\mathfrak{E} = (E, \phi)$ be a rank r Higgs bundle on X , and let $\Delta(E)$ be the characteristic class

$$\Delta(E) = c_2(E) - \frac{r-1}{2r}c_1(E)^2 = \frac{1}{2r}c_2(E \otimes E^*).$$

We shall denote by $\mathfrak{E} \otimes \mathfrak{E}^*$ the Higgs bundle $(E \otimes E^*, \psi)$ where ψ is obtained by coupling ϕ and ϕ^* in the usual way.

$\mathfrak{Gr}_s(\mathfrak{E})$ denotes as before the Grassmannian of Higgs quotients of \mathfrak{E} , while $\mathfrak{Q}_s = (Q_s, \Phi_s)$ is the universal quotient Higgs bundle on $\mathfrak{Gr}_s(\mathfrak{E})$. Denoting by $\pi_s: \mathbb{P}Q_s \rightarrow X$ the projection onto X , one defines the classes $\lambda_{s,\mathfrak{E}} \in N^1(\mathbb{P}Q_s)$ as

$$\lambda_{s,\mathfrak{E}} = c_1(\mathcal{O}_{\mathbb{P}Q_s}(1)) - \frac{1}{r}\pi_s^*c_1(E).$$

We shall need a result similar to Proposition 2.1 in the case of Higgs bundles. We prove a weaker version which is sufficient to our purpose, but very likely this result can be strengthened.

Definition 4.1. *A Higgs bundle \mathfrak{E} is said to be Higgs-nef if the line bundle $\mathcal{O}_{\mathbb{P}Q_s}(1)$ is nef for every $s = 1, \dots, r-1$ (or, in other terms, if all bundles Q_s are nef). If both \mathfrak{E} and \mathfrak{E}^* are Higgs-nef, \mathfrak{E} is said to be Higgs-numerically flat.*

Lemma 4.2. *If \mathfrak{E} is semistable and Higgs-numerically flat with $c_1(E) = 0$ and there exists a section $\sigma: X \hookrightarrow \mathfrak{Gr}_1(\mathfrak{E})$ of the Higgs Grassmannian $\rho_1: \mathfrak{Gr}_1(\mathfrak{E}) \rightarrow X$, then all Chern classes of E vanish.*

Proof. We first prove the statement when X is a surface. By the definition of Higgs-nefness, all the universal bundles Q_s on $\mathfrak{Gr}_s(\mathfrak{E})$ are nef. Let us consider the exact sequence

$$0 \rightarrow S_1 \rightarrow \rho_1^*E \rightarrow Q_1 \rightarrow 0.$$

Under the identification $\mathfrak{Gr}_{r-1}(\mathfrak{E}^*) \simeq \mathfrak{Gr}_1(\mathfrak{E})$ the bundle S_1 gets identified with Q_{r-1}^* . If we take σ^* in the above sequence we have

$$0 \rightarrow \sigma^*(S_1) \rightarrow E \rightarrow \sigma^*(Q_1) \rightarrow 0.$$

Now $\sigma^*(Q_1)$ and $\sigma^*(S_1)^*$ are nef. By Theorem 2.5 of [2] (but since we are on a surface one can also easily prove this by direct computation) we have $c_1(\sigma^*(S_1)^*)^2 - c_2(\sigma^*(S_1)^*) \geq 0$. Since $c_1(E) = 0$ one has $c_2(E) \leq 0$, which together with the Bogomolov inequality $c_2(E) \geq 0$ yields $c_2(E) = 0$ as desired.

If $n = \dim X > 2$, taking $m \gg 0$ and a smooth hypersurface Y in the linear series $|mH|$, the restriction of \mathfrak{E} to Y is still Higgs-semistable and Higgs-numerically flat with vanishing first Chern class, and $\mathfrak{Gr}_1(\mathfrak{E}|_Y)$ has a section. We can iterate this until we get a surface Z . Then we have $c_2(E|_Z) = 0$ and therefore $c_2(E) \cdot H^{n-2} = 0$. Then by [9, Thm. 2] \mathfrak{E} is an extension of stable Higgs bundles with vanishing Chern classes, so that the Chern classes of E vanish as well. \square

Proof of Theorem 1.3. We first prove that i) implies ii), dividing the proof into steps.

Step 1. Let $f: C \rightarrow X$ be a morphism, where C is a smooth projective curve. Then $f^*\mathfrak{E}$ is semistable. Indeed the definitions of the bundles Q_s and of the classes $\lambda_{s,\mathfrak{E}}$ are functorial, so that $\lambda_{s,f^*\mathfrak{E}} = \tilde{f}^*\lambda_{s,\mathfrak{E}}$, where $\tilde{f}: \mathbb{P}Q'_s \rightarrow \mathbb{P}Q_s$ is the map induced by f (here Q'_s is the universal quotient Higgs bundle of f^*E). This implies that the classes $\lambda_{s,f^*\mathfrak{E}}$ are nef. By Theorem 1.2 $f^*\mathfrak{E}$ is semistable.

Step 2. We prove that \mathfrak{E} is semistable. Indeed by first restricting to the generic divisor in $|mH|$ for m big enough, and then iterating, we may assume that X is a surface. Applying the previous step to a generic curve in $|mH|$, again for m big enough, we obtain that $\mathfrak{E}|_{mH}$ is semistable, and then \mathfrak{E} is semistable.

Step 3. We prove that $\mathfrak{F} = \mathfrak{E} \otimes \mathfrak{E}^*$ is Higgs-numerically flat. Since \mathfrak{F} is isomorphic to its dual, the point is to prove that it is Higgs-nef. Fix s and let $\bar{f}: C \rightarrow \mathbb{P}Q_s(\mathfrak{F})$ be a finite morphism, C being a connected smooth curve. By Step 1, if $f = \pi_s \circ \bar{f}$, then $f^*\mathfrak{E}$ is semistable, and hence $f^*\mathfrak{F}$ is semistable so that $\lambda_{s,f^*\mathfrak{E}}$ is a nef class. Since $c_1(F) = c_1(E \otimes E^*) = 0$, the class $\lambda_{s,f^*\mathfrak{E}}$ equals $c_1(\mathcal{O}_{\mathbb{P}f^*(Q_s(\mathfrak{F}))}(1))$. Let $\tau: \bar{f}(C) \rightarrow \mathbb{P}Q_s(\mathfrak{F})$ be the section induced by \bar{f} ; one has

$$\deg \tau^* \mathcal{O}_{\mathbb{P}f^*(Q_s(\mathfrak{F}))}(1) = [\bar{f}(C)] \cdot c_1(\mathcal{O}_{\mathbb{P}(Q_s(\mathfrak{F}))}(1))$$

so that $Q_s(\mathfrak{F})$ is nef, i.e., \mathfrak{E} is Higgs-nef.

Step 4. We show that $\Delta(E) = 0$. Since \mathfrak{F} is Higgs-numerically flat and semistable as a Higgs bundle and the Higgs Grassmannian $\mathfrak{Gr}_1(\mathfrak{F})$ has a section induced by the evaluation morphism $\mathfrak{F} \rightarrow \mathcal{O}_X$, all Chern classes of $E \otimes E^*$ vanish by Lemma 4.2, whence the claim.

Now we prove the converse statement, again dividing it into steps.

Step 1. The present hypotheses imply that \mathfrak{F} is semistable, and $c_2(E \otimes E^*) = 0$, while of course $c_1(E \otimes E^*) = 0$. By [9, Thm. 2] there is a filtration in Higgs subbundles

$$0 = \mathfrak{F}_0 \subset \mathfrak{F}_1 \subset \cdots \subset \mathfrak{F}_s = \mathfrak{F}$$

such that every quotient $\mathfrak{F}_i/\mathfrak{F}_{i-1}$ is stable and has vanishing Chern classes. Again by results contained in [9] we know that each quotient $\mathfrak{G}_i = \mathfrak{F}_i/\mathfrak{F}_{i-1}$ admits a Hermitian-Yang-Mills metric. Let Ω_i be the curvature of the associated Chern connection. Since $c_1(G_i) = c_2(G_i) = 0$, we have

$$0 = \int_X \text{tr}(\Omega_i \wedge \Omega_i) \cdot H^{n-2} = \gamma_1 \|\Omega_i\|^2 - \gamma_2 \|\Lambda \Omega_i\|^2 = \gamma_1 \|\Omega_i\|^2$$

for some positive constants γ_1, γ_2 , so that the Chern connection of G_i is flat.

Step 2. For a fixed s with $0 < s < r$, let $\bar{f}: C \rightarrow \mathbb{P}Q_s$ be a finite morphism, where C is a smooth irreducible projective curve. Let $f: C \rightarrow X$ be the composition of \bar{f} with the projection $\mathbb{P}Q_s \rightarrow X$. We show that the Higgs bundle $f^*\mathfrak{E}$ is semistable. Indeed the Higgs bundle $f^*\mathfrak{F}$ is filtered by the Higgs bundles $f^*\mathfrak{F}_i$, and the pullbacks $f^*(\mathfrak{F}_i/\mathfrak{F}_{i-1}) \simeq f^*\mathfrak{F}_i/f^*\mathfrak{F}_{i-1}$ carry flat unitary connections, hence they are polystable (again [9, Thm. 1]). Moreover they all have degree zero. As a consequence, $f^*\mathfrak{F}$ is semistable, and $f^*\mathfrak{E}$ is semistable as well.

Step 3. By Theorem 1.2, the classes $\tilde{\lambda}_{s,\mathfrak{E}}$ in $N^1(\mathbb{P}Q'_s)$ are nef. This implies that for any irreducible curve $C' \subset \mathbb{P}Q_s$ one has $[C'] \cdot \lambda_{s,\mathfrak{E}} \geq 0$, i.e., it implies that all classes $\lambda_{s,\mathfrak{E}}$ are nef. Indeed if C is the normalization of C' we can apply the previous constructions to C .

This concludes the proof of Theorem 1.3.

Theorem 1.3 has an immediate Corollary.

Corollary 4.3. *A semistable Higgs bundle $\mathfrak{E} = (E, \phi)$ on an n -dimensional projective polarized complex manifold (X, H) such that $c_1(E) \cdot H^{n-1} = \text{ch}_2(E) \cdot H^{n-2} = 0$ is Higgs-numerically flat.*

Proof. Again by [9, Thm. 2] all Chern classes of E vanish. So $\Delta(E) = 0$, and by Theorem 1.3, all classes $\lambda_{s,\mathfrak{E}}$ are nef. But since $c_1(E) = 0$ this implies that all bundles Q_s are nef,

i.e., \mathfrak{E} is Higgs-nef. Applying the same argument to the dual Higgs bundle \mathfrak{E}^* one obtains the claim. \square

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